HIP ORTHOSIS SERVOMOTOR TORQUE REQUIREMENTS

James L. Farrell
VIGIL, Inc.
Severna Park, MD.

ABSTRACT

A complex structural analysis model [Refs. 1,2], previously programmed and verified by successful comparison vs an independent approach [Ref. 3], has been applied to gait dynamics in the sagittal plane. The ultimate aim of this analysis/program is application to servomotor-powered hip orthosis for paraplegics. An interim goal, covered herein, is the determination of representative time histories for torques at hip and knee joints. Adaptation of the analysis is described and interpreted in light of related efforts. Results are provided for an empirical gait pattern adapted from Ref. 4, after which plans for future application to complete control system design and validation are discussed.

BACKGROUND

In the mid-1980s the Volunteers for Medical Engineering (VME) at Westinghouse Defense & Electronics Systems (Baltimore), with direction from Dr. Arthur Siebens of Johns Hopkins Medical School, began investigating the possibility of using servomotors for a paraplegics' powered hip orthosis. Though other means of providing ambulatory capability are studied elsewhere, ground rules for this development were fundamentally different. No functional electrical stimulation (FES; Refs. 5,6) is contemplated here, and the complete set of provisions to be added must eventually be portable.

Immediately the overall effort branches into various directions, including compilation of candidate control approaches, feasibility analysis from the standpoint of energy consumption/transfer, a survey of available motors, power supplies, etc. As a prerequisite toward design and validation of the control scheme this presentation addresses the dynamics of the 7-member structure shown in Fig. 1 (similar to the 5-member model used in Ref. 7). Specifically, the outcome is a is a solution for the moment relationships thus: The model characterizes gait dynamics in the sagittal plane. Any specified individual can be structurally represented by a modest number of parameters (e.g., weight, length, etc. for each of the structural members). Detailed time histories for hip and knee angles were adapted from data in Ref. 4. Given that angular time history, there clearly must be a corresponding time history of torques, to be supplied at each hip and knee joint, that will produce the gait. This analysis determines the torque specification, via a computer program based on complex formalisms of Refs. 1,2.

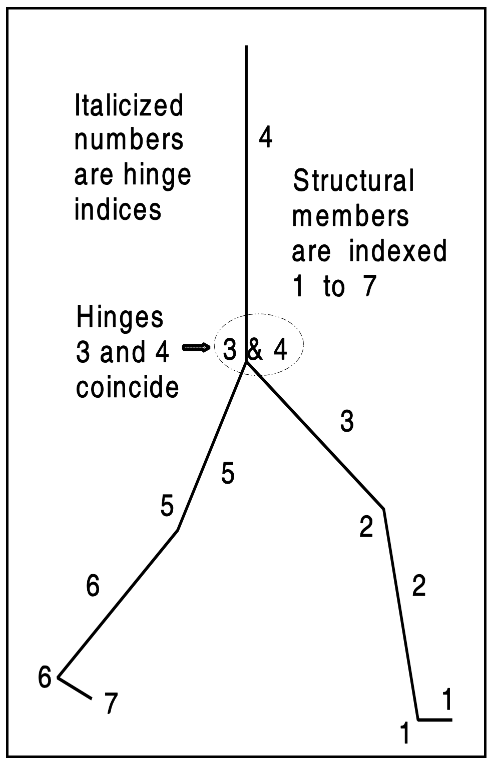

Figure 1: *7-member Structure*

Some further comments at this point will clarify the perspective:

1) Similarity of the 7-member structure to the model used in Ref. 7 is a coincidence but, in view of the application, not a very surprising one. A more interesting parallel between this analysis and Ref. 7 lies in the fact that, given any specified empirical time history for the angles, the corresponding torques should be in agreement. No effort is made here to demonstrate that, but its verification would make a worthwhile project.

2) Analogy between the approach in Ref. 7 (Lagrangian analysis) with the methods described herein produces another coincidence, and also a fundamental difference. The coincidence is that, in Ref. 3, an independent Lagrangian model was used to validate the program based on the rigid-member assembly analyzed by the dynamical formalism adopted from Refs. 1 and 2. A basic difference, however, is that the Lagrangian analysis of Ref. 3 used an elastic continuum (not applicable to gait dynamics analysis performed here nor in Ref. 7) as the independent model for comparison.

3) Although the program described here is intended for further use as the means of future control law implementation, a development using the Lagrangian model of Ref. 7 could also produce a mechanization approach.

In any case the path to be followed is clear, and a preliminary analysis can begin.

INTRODUCTION

The formalism to be used will include internal and external moments, with the latter subdivided into gravitational and nongravitational contributors. Because this formalism is quite complex, a preparatory discussion with familiar examples seems appropriate. The first objective here will be to raise and resolve a conceptual subtlety: External moments appearing in the applicable dynamic equations do not involve weight *per se*. Gravity is a *distributed* force which by itself would cause all points on any structure to accelerate downward *in tandem,* thus exerting a negligible influence on the rotational dynamics here. Mathematically this is shown in Eq. (19) of Ref. 1; only the gravity *gradient* remained from that derivation, and the gradients are too small to be significant in a 1-g force field. Immediately this raises a paradox to those unfamiliar with that concept: Since gravity provides that 1-g force field, how can it be ignored in this development? The answer is that it is not ignored; it enters in the upward restoring force that *counteracts* the weight. In marked contrast to that (distributed) weight, restoring forces are essentially concentrated at the point of contact --- and the effect of *that* force in this application produces a dominant moment that externally supplied orthosis motors must counteract.

Figure 2 shows the first introductory example chosen for illustration. With g denoting gravity and m as total mass, total weight of both members is mg. A ground reaction force $\mathbf{F}_1'$ with magnitude mg acts directly on the first member. Consider a motor with its stator attached to that vertical member and its rotor attached to the second member. A student, asked what motor torque would hold member #2 horizontal, would likely answer: $m_2 g L_2$.

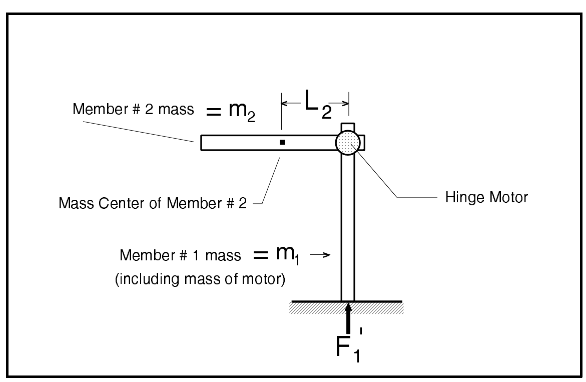

Figure 2: *Static 2-member Structure*

The motor torque just stated was determined from a basic concept (support of a structural member's weight at a known lever arm distance) which, for simple problems, can produce quick answers. In preparation for more complex structural interactions, however, greater sophistication is in order. As a first step, a two-member analysis from Ref. 8 will be adapted to Fig. 2. Although the analysis in that reference was for a free-free structure such as a satellite, it allowed for the addition of direct external forces (in this example $\mathbf{F}_1'$ and $\mathbf{F}_2'$) and external torques (in this example $\mathbf{T}_1'$ and $\mathbf{T}_2'$) acting on each member; thus the analysis can readily be applied to Fig. 2. Only nongravitational external effects are to be added since, as mentioned in a preceding paragraph, a small gradient is the sole effect of the gravitational field – and here that gradient can be neglected. For the static case (zero angular rates, zero angular accelerations) with no constraint torques nor reaction torques at the hinge[§] Eqs. (10 a) and (10 b) of Ref. 8 reduce to

$$\begin{aligned} \mathbf{T}_1' &= \frac{m_1}{m}\, \boldsymbol{L}_1 \times \mathbf{F}_2' - \frac{m_2}{m}\, \boldsymbol{L}_1 \times \mathbf{F}_1' \\ \mathbf{T}_2' &= \frac{m_2}{m}\, \boldsymbol{L}_2 \times \mathbf{F}_1' - \frac{m_1}{m}\, \boldsymbol{L}_2 \times \mathbf{F}_2' \end{aligned} \qquad (1)$$

In this first example both $\mathbf{F}_2'$ and $\boldsymbol{L}_1$ are zero, so that $\mathbf{T}_1'$ is zero and $|\mathbf{T}_2'| = m_2 g L_2$. When viewed in this way, the issue of weight influences the static solution by affecting the magnitude of $\mathbf{F}_1' = mg$ and through the ratio m_2/m; this perspective differs somewhat from the basic conceptual solution described in the preceding paragraph. Motor torque is now viewed as offsetting the effect of a concentrated ground reaction $\mathbf{F}_1'$ (rather than a distributed weight). Note also that another clockwise moment $m_2 g L_2$ must be supplied at the base of member # 1, to prevent collapse of the whole assembly.

A somewhat more relevant example can now be described wherein a junction connecting two lightweight masses m_1 and m_2 is joined by an additional heavy vertically oriented member with mass m_3 (Fig. 3). Stators of two separate motors are both fastened at the bottom of member # 3; one rotor is attached to m_1 and the other to m_2. These will be referred to as the "right" and "left" respectively, despite adoption of a sagittal planar approximation to describe all motion. The left leg places only a modest demand on its motor but, since $\mathbf{F}_1'$ counterbalances total mass $m_1 + m_2 + m_3$, the motor on the "right" (facing outward from the page) must counteract the moment of the heavy mass, acting through lever arm $\boldsymbol{L}$. A basic analogy with, and a basic difference from, normal ambulation will now be explained:

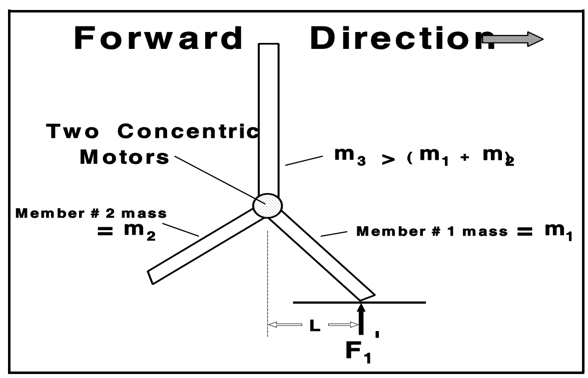

Figure 3: *3-member Structure*

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§ Constraint torques arise from having less than three rotational degrees of freedom at a hinge. These can be considered nonexistent merely by defining hinges to allow three-dimensional rotations; at the same time, there will be no rotations out of the sagittal plane – since initial conditions and forcing functions are zero for all out-of-plane values throughout this paper.
Reaction torques in Ref. 8 are passive components such as elastic restoring or dissipative damping torques; again, these do not play any significant role in this analysis.

Figure 3 crudely depicts conditions after rear-foot liftoff, with the line of action for $\mathbf{F}_1'$ passing through the trailing edge of member #1 (and of course with no torque supplied from the floor to maintain static conditions). A backward fall is prevented by forward momentum of the structure plus, for normal ambulation, a "motor" torque generated at the forward ankle joint. The latter contribution will be absent from the powered orthosis assembly described herein – *a major departure from normal ambulation*. With these clarifications, the simplified structures of Figs. 2 and 3 are no longer needed; further conceptual expansion will again refer to Fig. 1.

BASIC CONCEPTS

This section is intended for introductory conceptual development only; preliminary and detailed analysis appear in APPENDIX A and B, respectively.

Figure 4 shows hip and knee time histories adapted from Ref. 4. If these are used to characterize the right leg, then companion histories for the left begin at abscissa 51, continuing to abscissa 100, jumping abruptly to abscissa 1, proceeding through abscissa 50. Unfortunately the adaptation of empirical data (e.g., to suit specifically selected patients) can entail substantial additional work. This refers not just to the acquisition and validation of a "correct" data set, nor to mundane scaling of graphical plots, but to more subtle implications. First, to keep the formulation manageable, there is a need to capitalize on the fortuitous finding in Ref. 4 that the curves for a given subject (patient) retain similar shapes for a wide range of speeds (number of steps per minute). Among other issues the characterization also calls for periodicity; any algorithm used to fit the observational data must have repetitive cycles. Specifically the ordinate at abscissa 1 must logically follow the ordinate at abscissa 100. That may seem trivially easy until similar requirements are imposed on the first two derivatives. Because of the relation between moments and second derivatives of angles, innocent-looking rotational time histories often give rise to large spurious torques (reminiscent of problems associated with differentiating a waveform with abrupt transitions --- twice)! Severity of these effects caused this development to be sidetracked for an extended period but, after numerous (futile) attempts at *ad hoc* solutions (e.g., adding more points at either or both ends of the data stream before fitting; shifting the cycle barrier time, etc.) the *periodic* cubic splines [Ref. 9] were found adequate to avoid further difficulty. Periodicity is especially important in this application because, without it, cubic splines obviate spurious torques everywhere *except* at the transition between consecutive cycles.

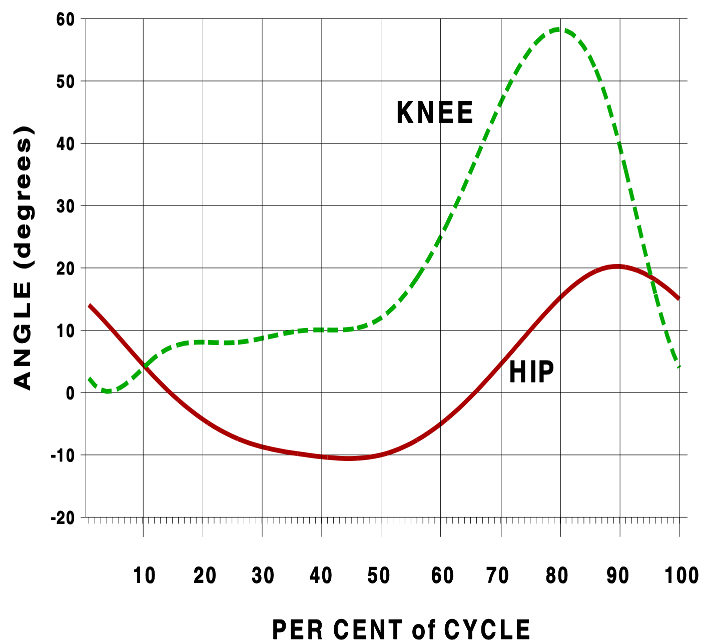

Figure 4: *Hip and knee angle histories*

No sooner was the spline-fit accommodation found than additional unexpected problems arose. One of these involved the requisite number of structural members; initially a simplified model was attempted, without the short segments from ankle/heel joint [§] to front instep. Questionable numerical results obtained with that 5-member model were finally diagnosed as violations of an obvious requirement: equal lengths for total vertical hip-joint-to-floor distances when both feet touch the floor. Once this realization set in, the entire analysis and program were reformulated. The two short segments were included, with angles at their joints "slaved" as follows: During the part of the gait cycle wherein both feet touch the floor, total hip-to-ankle/heel vertical projections of left and right leg were readily determined from instantaneous values of hip and knee angles. The longer of the two is given a short segment with horizontal orientation (flat on the floor); the shorter is assigned the ankle/heel joint angle value necessary to equalize total hip-to-floor vertical distance. For the part of the gait cycle wherein only one foot touches the floor, angles at the ankle/heel joints vary linearly with time, between boundary values (the entire sequence was given a "reasonableness" test, whereby the structure in Fig. 1 was animated to walk across the computer screen). For line-of-action, the force at the point of floor contact is concentrated at either 1) the center of the member (#1 or #7) when flat, or 2) the forward instep when not flat on the floor. This characterization affects the handling of external forces in the adopted formalism; the lever arm for external force at the point of contact is either zero (flat-on-the-floor segment) or half the length of that short segment (force concentrated at forward instep).

Another major issue causing temporary sidetracking of this effort was a more subtle discovery, regarding conditions prior to the state depicted in Fig. 3. Initially it seemed reasonable to assume that, during the part of the gait cycle when both feet touch the floor, a plausible characterization could be assigned for the time history of percent weight on each foot. This is not permissible; that fraction is implicit in the hip and knee angle time history. For proof, choose a stationary reference point to coincide with the point of contact for the right foot; note that the total angular momentum with respect to that point, and the time derivative of that angular momentum, are uniquely determined from the angle time histories. Note also that this angular momentum's time derivative must equal the sum of torques about that point and, with zero lever arm for frictional forces, the sum of torques consists of *a*) the moment of the left foot's restoring force and *b*) products of each member's weight multiplied by the horizontal lever arm from the point of contact, in a direction consistent with the preceding paragraph. This relation is used to solve for the restoring force, and hence the instantaneous fraction of weight on each foot. The analysis was sufficiently complex to warrant a separate development (APPENDIX C). It is important to recognize that the question raised by this vertical force analysis concerns the issue of distribution, *i.e.*, what fraction of the total is associated with each foot. The possibility of two *opposing* forces never arises. That differs markedly from the situation with horizontal forces. As the forward foot touches the floor, it acts as a brake on the forward horizontal force provided by pushoff from the rear foot. The horizontal forces cannot be analyzed by the methods of Appendix C – which prompted still another side investigation, which will now be discussed.

[§] To avoid adding still more small members, the distance from ankle to heel is ignored here; that displacement does not significantly affect torques at hip and knee joints.

Appendix C solves for distribution of vertical forces by equating net torque with the time derivative of angular momentum, using an inertial (stationary) reference. The only stationary points are on the floor and, for horizontal point-of-contact forces, all lever arm lengths are zero. There still is a relation equating the time derivative of linear momentum with net force, but that alone does not uniquely define the horizontal contact forces. Unlike the vertical case, then, the horizontal force analysis permits an added condition to be imposed. One way to describe the ramifications is to denote the fraction of the weight on member #1 (the "right" foot in the sagittal plane) as f and to write the upward vertical forces supporting members #1 and #7 (the "left" foot) as

$$\rho_{1y} = f \times [(\textit{total weight}) + (\textit{upward component of total linear momentum rate})]$$

(2)

$$\rho_{7y} = (1 - f) \times [(\textit{total weight}) + (\textit{upward component of total linear momentum rate})]$$

For horizontal components, member #7 pushes back against the floor – thus generating a forward static frictional force on itself (with coefficient denoted by ζ). Member #1 partially counteracts that frictional force, while the algebraic resultant drives the forward momentum:

$$\rho_{7x} = \zeta \times \rho_{7y}$$

(3)

$$\rho_{1x} = - \rho_{7x} + (\textit{forward component of total linear momentum rate})$$

In both cases the pair of forces will conform to the Newtonian sum. Vertical forces complement each other whereas, for horizontal forces, one has a retarding effect on the other. Since the horizontal force solution is not unique, equal amounts of opposing forces could be superimposed with no effect (their lever arm lengths are zero). No attempt is made here to specify the "best" horizontal force level, but future evaluations might investigate possible minimization of total torque or energy. If such minimizations are done somewhat unconsciously or involuntarily, that would constitute another significant departure from normal ambulation.

The last unexpected complication came from the data in Ref. 4. The previously mentioned "reasonableness" test (wherein an animated structure of Fig. 1 walked across the computer screen) looked very strange with the data unmodified. This was probably due to different lengths (hip-to-knee, knee-to-ankle/heel) in comparison to values used here. Changes were introduced into the angle histories until the animation appeared normal.

With the foregoing discussion in mind, attention can be shifted to the adopted model of gait dynamics in the sagittal plane, applied to the 7-member structure in Fig. 1, using concepts and notation from Ref. 1.

APPROACH

To characterize the motion of the coupled rigid bodies in Fig. 1, the rotational dynamics can first be expressed as a set of equations in the usual form,

$$\mathbf{I}\,\dot{\omega} + \omega \times \mathbf{I}\,\omega = \mathbf{T} \tag{4}$$

wherein $\mathbf{I}$, ω, and $\mathbf{T}$ denote inertia tensor, absolute angular rate vector, and total torque vector, respectively. Since this Euler relation holds for each of the N members of the structure, it can be reinterpreted as a $3N$-dimensional equation; ω and $\mathbf{T}$ therefore have $3N$ components, while $\mathbf{I}$ is a $3N$-by-$3N$ matrix made up of 3x3 partitions on the diagonal (off-diagonal partitions are 3x3 null matrices). $\mathbf{T}$ consists of external torques, internal torques, plus moments of internal and direct (*i.e.* , not arising from other structural members' interaction) external forces. Since the internal forces are generally unknown and not of primary interest in themselves, it is desirable to replace them by equivalent quantities obtained from Newton's laws. Consequently the internal forces are reëxpressed in terms of external and d'Alembert forces; the motion of the composite structure mass center is then eliminated from the equations. As a result, the d'Alembert forces can be defined by second derivatives of position vectors relative to this composite mass center. Through the dynamical formalism, moments of these d'Alembert forces are written in a convenient computational arrangement whereby

1) part of the centripetal component is included in a constituent $\mathbf{Q}$ of $\mathbf{T}$;

2) the remainder of the centripetal component is taken into account through replacing $\mathbf{I}$ in the second term of Eq. (1) by a constant augmented inertia matrix $\mathbf{K}$;

3) the tangential component (associated with $\dot{\omega}$) is taken into account by further augmenting the inertia matrix to a value $\mathbf{K} + \Psi$, where Ψ varies as a known function of the structural members' attitude matrices.

Eq. (15) of Ref. 1 is a rewritten form of Eq. (1), with internal torques and internal force moments reconstituted (after considerable mathematical development) into $\mathbf{e}^{\mathrm{T}} \odot \mathbf{S}\pounds$ and $\mathbf{PF}$, respectively, while external torques and moments of external (as previously explained, direct and nongravitational) forces are combined in $\mathbf{L}$:

$$(\mathbf{K} + \Psi)\,\dot{\omega} + \omega \times \mathbf{K}\,\omega = \mathbf{L} - \mathbf{PF} + \mathbf{e}^{\mathrm{T}} \odot \mathbf{S}\pounds + \mathbf{Q} \tag{5}$$

This equation is reëxpressed for the application at hand (APPENDIX B) and programmed in FORTRAN (APPENDIX D). Results are now given for representative parameter values.

PROGRAM DESCRIPTION and RESULTS

CONCLUSIONS and FUTURE PLANS [MUCH remains to be done]

NEAR TERM:

- Accommodate sudden torque demand @ "heel strike" - actually flatfoot strike
- Vary parameters (weights [incl. motor wt.], lengths, friction, etc.)
- dx/dt = Ax + Bu & complete control specification
- invite critique ('intimately related' stmt ~ ctr of JI-53 paper ?)

FUTURE:

- extension: crutches & on-line adapt to *patient's adaptation*
- ~~general curve fit for any empirical data~~ ⇐ DONE
- generalized condit: Tilt member #4, read time-varying β_4 from synchro, ground slope
- Energy computations
- Springs
- Start-up conditions for all angles and rates: violates "separability" condition.
- Ski poles --> Frontal as well as sagittal plane [2-D as check case]
- structural asymmetry
- sit/stand transition
- simulate motor and solve for rotational history

LIMITATION (or possible future refinement):

A backward fall is prevented by forward momentum of the structure plus, for normal ambulation, a "motor" torque generated at the forward ankle joint. The latter contribution will be absent from the powered orthosis assembly described herein – *a major departure from normal ambulation*.

ACKNOWLEDGEMENTS

Without guidance from Dr. Arthur Siebens of Johns Hopkins Medical School and VME founder Mr. John Staehlin, the work described herein would be nonexistent. Without the dynamical formalism from Refs. 1 and 2, this author would never have known how to perform this analysis (nor half of that in Ref. 3). The author is indebted also to Mr. David A. Hedland for many insightful conversations in regard to the hip orthosis development.

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APPENDIX A:
FUNDAMENTAL KINEMATICAL RELATIONSHIPS

With the off-vertical tilt angle of the trunk (member #4 in Fig. 1) denoted as β_4 , the tilt β_3 of member #3 is simply the sum of β_4 plus the instantaneous right hip angle; β_2 is β_3 minus the right knee angle. For β_5 and β_6 , simply substitute hip and knee angles with a half-cycle offset. Angles β_1 and β_7 were slaved as just described. Angular rates and accelerations are simply formed by subtracting the time-centered angles and dividing by the appropriate time-intervals (discussed later in this APPENDIX).

Given values for these instantaneous angles, rates, and accelerations, it is a straightforward task to express the direction for each member's unit vector (with positive direction defined from lowest to highest point on the member; e.g., β_2 and β_3 are positive in Fig. 1) as well as the first and second derivatives for each vector, in the previously defined reference coordinate frame. The z-components of course vanish, and

$$\boldsymbol{u_i} = \begin{bmatrix} -\sin\beta_i \\ \cos\beta_i \end{bmatrix} \tag{A-1}$$

thus

$$\dot{\boldsymbol{u}}_{\boldsymbol{i}} = -\dot{\beta}_i \begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} \equiv -\omega_i \begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} \tag{A-2}$$

and therefore

$$\ddot{\boldsymbol{u}}_{\boldsymbol{i}} = -\dot{\omega}_i \begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} + \omega_i^2 \begin{bmatrix} \sin\beta_i \\ -\cos\beta_i \end{bmatrix} \tag{A-3}$$

The d'Alembert forces are derived by applying the latter expression to each member of length l_i. When the bottom of member #1 is inertially fixed, the computation of absolute second derivatives at each member's mass center will conform to the following sequence:

$$\begin{aligned}
\ddot{\boldsymbol{R}}_{01} &= \frac{1}{2}\, l_1\, \ddot{\boldsymbol{u}}_1 \\
\ddot{\boldsymbol{R}}_{02} &= \ddot{\boldsymbol{R}}_{01} + \frac{1}{2}\,(l_1\, \ddot{\boldsymbol{u}}_1 + l_2\, \ddot{\boldsymbol{u}}_2) \\
\ddot{\boldsymbol{R}}_{03} &= \ddot{\boldsymbol{R}}_{02} + \frac{1}{2}\,(l_2\, \ddot{\boldsymbol{u}}_2 + l_3\, \ddot{\boldsymbol{u}}_3) \\
\ddot{\boldsymbol{R}}_{04} &= \ddot{\boldsymbol{R}}_{03} + \frac{1}{2}\,(l_3\, \ddot{\boldsymbol{u}}_3 + l_4\, \ddot{\boldsymbol{u}}_4) \\
\ddot{\boldsymbol{R}}_{05} &= \ddot{\boldsymbol{R}}_{03} + \frac{1}{2}\,(l_3\, \ddot{\boldsymbol{u}}_3 - l_5\, \ddot{\boldsymbol{u}}_5) \\
\ddot{\boldsymbol{R}}_{06} &= \ddot{\boldsymbol{R}}_{05} - \frac{1}{2}\,(l_5\, \ddot{\boldsymbol{u}}_5 + l_6\, \ddot{\boldsymbol{u}}_6) \\
\ddot{\boldsymbol{R}}_{07} &= \ddot{\boldsymbol{R}}_{06} - \frac{1}{2}\,(l_6\, \ddot{\boldsymbol{u}}_6 + l_7\, \ddot{\boldsymbol{u}}_7)
\end{aligned} \tag{A-4}$$

and, when the bottom of member #7 is inertially fixed, the sequence becomes

$$
\begin{aligned}
\ddot{\boldsymbol{R}}_{07} &= \frac{1}{2}\, l_7\, \ddot{\mathbf{u}}_7 \\
\ddot{\boldsymbol{R}}_{06} &= \ddot{\boldsymbol{R}}_{07} + \frac{1}{2}\,(l_7\, \ddot{\mathbf{u}}_7 + l_6\, \ddot{\mathbf{u}}_6) \\
\ddot{\boldsymbol{R}}_{05} &= \ddot{\boldsymbol{R}}_{06} + \frac{1}{2}\,(l_6\, \ddot{\mathbf{u}}_6 + l_5\, \ddot{\mathbf{u}}_5) \\
\ddot{\boldsymbol{R}}_{04} &= \ddot{\boldsymbol{R}}_{05} + \frac{1}{2}\,(l_5\, \ddot{\mathbf{u}}_5 + l_4\, \ddot{\mathbf{u}}_4) \\
\ddot{\boldsymbol{R}}_{03} &= \ddot{\boldsymbol{R}}_{05} + \frac{1}{2}\,(l_5\, \ddot{\mathbf{u}}_5 - l_3\, \ddot{\mathbf{u}}_3) \\
\ddot{\boldsymbol{R}}_{02} &= \ddot{\boldsymbol{R}}_{03} - \frac{1}{2}\,(l_3\, \ddot{\mathbf{u}}_3 + l_2\, \ddot{\mathbf{u}}_2) \\
\ddot{\boldsymbol{R}}_{01} &= \ddot{\boldsymbol{R}}_{02} - \frac{1}{2}\,(l_2\, \ddot{\mathbf{u}}_2 + l_1\, \ddot{\mathbf{u}}_1)
\end{aligned}
\tag{A-5}
$$

A mass-weighted sum of the individual vectors forms the composite mass center, related to external forces by a Newtonian law. That law is given by Eqs. (B-23) in the next APPENDIX. As this is by necessity expressed in the stable reference frame, the corresponding force components expressed in the coordinate frames of members #1 and #7 are given by Eq. (B-24) (for substitution into the expression for moments of external forces, also explained in the next APPENDIX).

As previously mentioned, Ref. 4 noted that hip and knee angle histories for a given subject (patient) have similar shapes for a wide range of speeds. Thus all time derivatives needed throughout this development can be found from normalized curves such as those in Fig. 4 for which, with 100 abcissa points, the angular rate centered about point i is

[{angle($i+1$) - angle($i-1$)} / 0.02] • [(number of cycles per minute) / 60] ,

with straightforward qualifications (e.g., two steps per gait cycle) and modifications (e.g., replacement of 101 by 1 when i = 100, or 0 by 100 when i = 1, etc.). All these relationships were of course used in programming subroutine KINEM of APPENDIX D.

APPENDIX B: ADAPTATION OF FORMALISM

In documenting the transition from the general formulation (Ref. 1) to this specific planar application, some complications have to be addressed at the outset:

1) Ref. 1 is quite general and complex. The formalism uses unusual conventions (such as matrices wherein elements can be vectors) and some higher order tensors.
2) The reference is obscure and not typeset (in fact, its weakest characteristic is its lack of visual clarity).
3) There are a few misprints (these are identified at the end of this APPENDIX for those interested).

Despite all complications, Ref. 1 is as indispensible to this analysis as it is challenging to grasp. Under the circumstances it was decided

- to avoid any duplication (such as description of the theory covered in Ref. 1),
- to provide FORTRAN listings (APPENDIX D) enabling others to use the program (and make parameter changes) without having to understand this analysis,
- to provide steps here for those readers interested in acquiring enough knowledge of Ref. 1 to pursue the theory,

and (therefore)

- to use various steps in Ref. 1 as a springboard for this analysis, which can now commence with the incidence matrix for the structure in Fig. 1:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B-1}$$

and, using the procedure described in Theorem 2 of Ref. 1, to compute the 6x7 left inverse,

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B-2}$$

The connection matrix **C** is formed through replacement of each entry in **S** by a 3x1 vector. All zeros are replaced by the null vector $\mathbf{0}$. Each nonzero element S_{ij} is replaced by the product of (S_{ij} itself) with (the corresponding mass-center-to-hinge vector for hinge #j and structural member #i). Each of these vectors is expressed in a coordinate frame wherein the y-axis $+\mathbf{1}_2$ is coincident with its upward length axis; thus in this application **C** has the simple form,

$$\mathbf{C} = \mathbf{1}_2 \begin{bmatrix} \frac{l_1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{l_2}{2} & \frac{l_2}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{l_3}{2} & \frac{l_3}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{l_4}{2} & \frac{l_4}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{l_5}{2} & \frac{l_5}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{l_6}{2} & \frac{l_6}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{l_7}{2} \end{bmatrix} \tag{B-3}$$

and the product **CT** is

$$\mathbf{D} = \mathbf{1}_2 \begin{bmatrix} \frac{l_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_2 & \frac{l_2}{2} & 0 & 0 & 0 & 0 & 0 \\ l_3 & l_3 & \frac{l_3}{2} & 0 & 0 & 0 & 0 \\ \frac{l_4}{2} & \frac{l_4}{2} & \frac{l_4}{2} & 0 & \frac{l_4}{2} & \frac{l_4}{2} & \frac{l_4}{2} \\ 0 & 0 & 0 & 0 & \frac{l_5}{2} & l_5 & l_5 \\ 0 & 0 & 0 & 0 & 0 & \frac{l_6}{2} & l_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{l_7}{2} \end{bmatrix} \tag{B-4}$$

The mass matrix corresponding to Eq. (5a) of Ref. 1 is simply

$$\mathbf{m} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_7 \end{bmatrix} \tag{B-5}$$

and, with $\mathbf{I}_{77}$ as the 7x7 identity matrix while $M = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7$ the normalized mass matrix corresponding to Eq. (5b) of Ref. 1 is

$$\mathbf{u} = \mathbf{I}_{77} - \frac{1}{M}\begin{bmatrix} m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 \\ m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 \\ m_3 & m_3 & m_3 & m_3 & m_3 & m_3 & m_3 \\ m_4 & m_4 & m_4 & m_4 & m_4 & m_4 & m_4 \\ m_5 & m_5 & m_5 & m_5 & m_5 & m_5 & m_5 \\ m_6 & m_6 & m_6 & m_6 & m_6 & m_6 & m_6 \\ m_7 & m_7 & m_7 & m_7 & m_7 & m_7 & m_7 \end{bmatrix} \tag{B-6}$$

The next step is to note, from Theorem 4 of Ref. 1, that

$$\mathbf{B} = \mathbf{D}\,\mathbf{u} \tag{B-7}$$

and that a modest amount of analysis can verify the identity

$$\mathbf{u}\,\mathbf{m}\,\mathbf{u}^{\mathsf{T}} \equiv \mathbf{m}\,\mathbf{u}^{\mathsf{T}} \tag{B-8}$$

Thus the matrix **J** (Theorem 6 of Ref. 1), can be written as

$$\mathbf{J} = \mathbf{D}\,\mathbf{m}\,\mathbf{u}^{\mathsf{T}}\,\mathbf{D}^{\mathsf{T}} = \mathbf{D}\,\mathbf{u}\,\mathbf{m}\,\mathbf{u}^{\mathsf{T}}\,\mathbf{D}^{\mathsf{T}} = \mathbf{B}\,\mathbf{m}\,\mathbf{B}^{\mathsf{T}} \tag{B-9}$$

and at this point various simplifications begin to emerge: First, every 3x3 partition of **J** has only one nonzero element, i.e., the coefficient of the dyadic $\mathbf{1}_2\mathbf{1}_2^{\mathsf{T}}$; thus every 3x3 partition of array $\boldsymbol{\phi}$ { Eq. (14a) of Ref. 1 } is a diagonal matrix of the form

$$\|\phi^i\| = \begin{bmatrix} J_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_i \end{bmatrix} \tag{B-10}$$

where the subscripted elements are obtained using Eq. (B-9) above with the simplifications noted; as a result, J_i is the (i,i)-element of the 7x7 matrix product $\mathbf{BmB}^{\mathsf{T}}$ obtained by forming Eq. (B-7) with the vector $\mathbf{1}_2$ in Eq. (B-4) ignored. With only z-axis rotation, addition of J_i to the principal moment I_{zz} for each corresponding inertia tensor forms a 7x7 augmented inertia matrix **K**. Also required is the transformation from j^{th} to i^{th} body axes introduced on page 46D.5 of Ref. 1; with β_i defining the angle of rotation from reference (+right, +up, +outward) to i^{th} body axes, this is expressible as a z-axis rotation through the angle $(\beta_i - \beta_j)$. Thus the matrix $\|\mathbf{P}_j^i\|$ in Eq. (12b) of Ref. 1 has the form,

$$\|\mathbf{P}_j^i\| = \begin{bmatrix} 0 & 0 & B_{ij2} \\ 0 & 0 & -B_{ij1} \\ P_{ij31} & P_{ij32} & 0 \end{bmatrix} ;$$

$$P_{ij31} = -B_{ij1}\sin(\beta_i-\beta_j) - B_{ij2}\cos(\beta_i-\beta_j)$$
$$P_{ij32} = B_{ij1}\cos(\beta_i-\beta_j) - B_{ij2}\sin(\beta_i-\beta_j) \tag{B-11}$$

and the (3,3) element of the matrix Ψ $(i \neq j)$ becomes

$$\Psi_{ij33} = M(P_{31}B_{ji2} - P_{32}B_{ji1}) \quad , \quad i \neq j \tag{B-12}$$

With all $B_{ji1} = 0$, B_{ji2} can be denoted simply as B_{ji} and, with Eq. (B-11) thus reduced to

$$\|\mathbf{P}_j^i\| = -B_{ij}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \cos(\beta_i - \beta_j) & \sin(\beta_i - \beta_j) & 0 \end{bmatrix} \tag{B-13}$$

then

$$\Psi_{ij33} = -MB_{ij}B_{ji}\cos(\beta_i - \beta_j) \quad , \quad i \neq j \tag{B-14}$$

This is the only *active* element in the matrix Ψ { Eq. (5) }, since the other elements in the third column are zero and all rotations occur about the z-axis in this application.

Planar rotation also ensures, since the augmented inertia matrix **K** is diagonal here, that ω is parallel to $\mathbf{K}\omega$; thus the second term of Eq. (5) vanishes. Note also that, with hinge torques covered by the next-to-last term of that equation, the only contribution to **L** will arise from moments of external forces at the point of contact, so that only member #1 or #7 can be affected. The notation v is chosen to represent the fraction of total positive (*i.e.*, forward horizontal or upward vertical) contact force, *e.g.*, f for member #1's or (1-f) for member #7's vertical reaction. Both a horizontal and a vertical positive force (denoted ρ_{jx} and ρ_{jy} respectively) would produce a counterclockwise moment about the extremity segment's mass center for positive β_j :

$$L_j = \nu[\rho_{jx}\ (^1/_2\, l_j \cos\beta_j) + \rho_{jy}\ (^1/_2\, l_j \sin\beta_j)] \quad , j = 1 \textit{ or } 7 \tag{B-15}$$

The means of reducing the formalism to a planar case can now be clarified. With the abovementioned terms missing, Eq. (5) is first replaced by

$$\mathbf{e}^{\mathrm{T}} \odot \mathbf{S}\,\pounds = (\mathbf{K} + \Psi)\,\omega + \mathbf{PF} - \mathbf{L} - \mathbf{Q} \tag{B-16}$$

and immediately we wish to extract only the z-component of each torque vector. Both sides of the above relation are premultiplied by a 7x21 matrix that can be partitioned as

$$\mathbf{Z} = \begin{bmatrix} Z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z \end{bmatrix} \tag{B-17}$$

wherein $\mathbf{0}$ is the 1x3 null row vector and

$$\mathbf{Z} = (0,0,1) = \mathbf{1}_3^{\mathrm{T}} \tag{B-18}$$

Now define the 6x1 vector of hinge torques

$$\boldsymbol{\lambda} = [\ \lambda_1\ \ \lambda_2\ \ \lambda_3\ \ \lambda_4\ \ \lambda_5\ \ \lambda_6\ ]^{\mathbf{T}} \tag{B-19}$$

and note that

$$\mathbf{Z}\,\mathbf{e}^{\mathrm{T}} \odot \mathbf{S}\mathcal{L} = \mathbf{S}\boldsymbol{\lambda} \tag{B-20}$$

Using the identity $\mathbf{TS} = \mathbf{I}_{66}$, a reduced form of Eq. (B-16) results;

$$\boldsymbol{\lambda} = \mathbf{T}\,\mathbf{Z}\{(\mathbf{K} + \Psi)\,\dot{\omega} + \mathbf{PF} - \mathbf{L} - \mathbf{Q}\} \tag{B-21}$$

Although the bracketed quantity in this latter expression is a 21x1 vector, **Z** extracts z-components throughout; thus only the matrix elements that produce z-axis torque components need be evaluated in each term. These z-components of each structural member's angular rate and angular acceleration are now written as the column vectors

$$\boldsymbol{w} = \left[\omega_{z1}\,\omega_{z2}\,\omega_{z3}\,\omega_{z4}\,\omega_{z5}\,\omega_{z6}\,\omega_{z7}\right]^{\mathrm{T}} \ ; \qquad \dot{\boldsymbol{w}} = \left[\dot{\omega}_{z1}\,\dot{\omega}_{z2}\,\dot{\omega}_{z3}\,\dot{\omega}_{z4}\,\dot{\omega}_{z5}\,\dot{\omega}_{z6}\,\dot{\omega}_{z7}\right]^{\mathrm{T}} \tag{B-22}$$

and it is noted from Fig. 1 that the only nonzero elements of the external force vector are the x- and y- components[§] for members #1 and #7. In terms of the total weight Mg, friction coefficient ζ, and acceleration $\ddot{\boldsymbol{R}}_{0i}$ at each member's mass center, expressed in the reference coordinate frame (+x = forward; +y = up), from Eqs. (2,3) these are

$$\rho_{1y} = f\ (Mg + \sum_1^7 m_i\,\ddot{\boldsymbol{R}}_{\mathbf{0}iy})\ ; \qquad \rho_{7y} = (1 - f\ )\ (Mg + \sum_1^7 m_i\,\ddot{\boldsymbol{R}}_{\mathbf{0}iy})\ ;$$
$$\rho_{7x} = \zeta\,\rho_{7y}\ . \qquad \rho_{1x} = \sum_1^7 m_i\,\ddot{\boldsymbol{R}}_{\mathbf{0}ix} - \rho_{7x} \tag{B-23}$$

where f is the instantaneous weight distribution factor; it is that fraction of external force on member #1. For this analysis that fraction is unity after 12% of the first half-cycle (Ref. 4, page 37), prior to which it builds from zero according to the schedule derived in APPENDIX C. Before premultiplication by elements of the matrix **P** these force components are transformed into axes of members #1 and #7 thus:

$$\begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} = \begin{bmatrix} \cos\beta_1 & \sin\beta_1 \\ -\sin\beta_1 & \cos\beta_1 \end{bmatrix} \begin{bmatrix} \rho_{1x} \\ \rho_{1y} \end{bmatrix} \ ; \qquad \begin{bmatrix} F_{7x} \\ F_{7y} \end{bmatrix} = \begin{bmatrix} \cos\beta_7 & \sin\beta_7 \\ -\sin\beta_7 & \cos\beta_7 \end{bmatrix} \begin{bmatrix} \rho_{7x} \\ \rho_{7y} \end{bmatrix} \tag{B-24}$$

To use these results in forming moments of external forces, it is helpful to use Eq. (B-13) and reduce the bottom row of $\|\mathbf{P}_j^i\|$ to

$$\boldsymbol{P}_{ij} = \mathbf{1}_3^{\mathrm{T}}\,\|\mathbf{P}_j^i\| = [\ -B_{ij}\cos(\beta_i - \beta_j) \quad -B_{ij}\sin(\beta_i - \beta_j) \quad 0\ ] \tag{B-25}$$

§ The x and y components of $\mathbf{F}$ are multiplied by elements in the bottom row of **P** in Eq. (B-13), for example, to produce z- components of torque. Note that, in (B-23), the front foot produces whatever forward force is required to counteract static friction at the rear foot *minus* the amount needed to maintain total forward acceleration. Vertical force overcomes weight while also maintaining vertical acceleration.

so that, when **ZPF** is partitioned as

$$\mathbf{ZPF} = \begin{bmatrix} P_{11} & P_{12} & - & - & - & - & - \\ P_{21} & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & P_{77} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} \tag{B-26}$$

only F_1 and F_7 are nonzero. Thus the i^{th} component of **ZPF** has the simple form,

$$(\mathbf{ZPF})_i = \boldsymbol{P}_{i1}\mathbf{F_1} + \boldsymbol{P}_{i7}\mathbf{F_7} \tag{B-27}$$

which produces simplification when combined with Eqs. (B-24) and (B-25), e.g., ρ_{1x} becomes multiplied by $[\cos\beta_1 \cos(\beta_i - \beta_1) - \sin\beta_1 \sin(\beta_i - \beta_1)] \equiv \cos\beta_i$. The final reduced expression is

$$(\mathbf{ZPF})_i = -B_{i1}(\rho_{1x}\cos\beta_i + \rho_{1y}\sin\beta_i) - B_{i7}(\rho_{7x}\cos\beta_i + \rho_{7y}\sin\beta_i) \tag{B-28}$$

The vector **Q** from Eq. (14 d) of Ref. 1 has a z-component that reduces to

$$Q_i = M \sum_{j \neq i} \omega_j^2 B_{ij} B_{ji} \sin(\beta_i - \beta_j) \tag{B-29}$$

The rotational dynamics subroutine (RODYN, APPENDIX D) was written in conformance to these expressions.

<u>MISPRINTS IN REF. 1</u>

Eq. (3): Delete M from left side.

Eq. (8): Replace $\mathfrak{F}$ by £ after S on right side.

Eq. (9 bis): Add £ after $\mathbf{e}^T \odot \mathbf{S}$ on right side.

Eq. (14 c): Subject to qualification "$i \neq j$"

p. 46D.6, rt. col: Replace "μ is a left inverse of $m\mu^T$" by : $\mu m \mu^T = m\mu^T$

Eq immediately thereafter: Replace "C" by "D" everywhere (6 places)

Eq (17a) bottom line: Tilde ~ over $B_{k\beta}^{i\alpha}$

APPENDIX C:
DERIVATION of WEIGHT DISTRIBUTION

Solution of the equations in the preceding APPENDIX requires determination of the fraction f of total weight on each foot. This is of course trivial when only one foot touches the floor, but not for the part of the gait cycle wherein both feet are down. As mentioned in APPENDIX A, that fraction is implicit in the hip and knee angle time history. With a stationary (inertial) reference point chosen to coincide with the right foot, the total angular momentum with respect to that point can be expressed in terms of linear mass densities α for each structural member, producing a sum of integrals having the form,

$$\mathbf{H} = \sum_{i=1}^{7} \alpha_i \int_0^{l_i} \mathbf{r}_i \times \dot{\mathbf{r}}_i \, ds \quad , \quad \dot{\mathbf{r}}_i = \omega_i \times \mathbf{r}_i \tag{C-1}$$

where {recall the positive upward convention from the discussion preceding Eq. (B-3) },

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{u}_1 s \\
\mathbf{r}_2 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 s \\
\mathbf{r}_3 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 l_2 + \mathbf{u}_3 s \\
\mathbf{r}_4 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 l_2 + \mathbf{u}_3 l_3 + \mathbf{u}_4 s \\
\mathbf{r}_5 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 l_2 + \mathbf{u}_3 l_3 - \mathbf{u}_5 s \\
\mathbf{r}_6 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 l_2 + \mathbf{u}_3 l_3 - \mathbf{u}_5 l_5 - \mathbf{u}_6 s \\
\mathbf{r}_7 &= \mathbf{u}_1 l_1 + \mathbf{u}_2 l_2 + \mathbf{u}_3 l_3 - \mathbf{u}_5 l_5 - \mathbf{u}_6 l_6 - \mathbf{u}_7 s
\end{aligned} \tag{C-2}$$

Direct differentiation of Eq. (C-1), followed by substitution of Eq. (C-2) and subsequent simplification, would obviously be unwieldly. Instead a computational algorithm is devised here, starting with the simplest terms from which the pattern becomes clear. For the first term in the sum for the derivative, [§]

$$\dot{\mathbf{H}}_1 = \alpha_1 \int_0^{l_1} s \, \mathbf{u}_1 \times \ddot{\mathbf{u}}_1 \, s \, ds \quad , \quad \ddot{\mathbf{u}}_1 = \dot{\omega}_1 \times \mathbf{u}_1 - \omega_1^2 \mathbf{u}_1 \tag{C-3}$$

so that the integrand reduces to

$$s^2 \, \mathbf{u}_1 \times (\dot{\omega}_1 \times \mathbf{u}_1) \, ds = s^2 \, \dot{\omega}_1 \, ds \tag{C-4}$$

and therefore

$$\dot{\mathbf{H}}_1 = \alpha_1 \int_0^{l_1} \dot{\omega}_1 \, s^2 \, ds = \frac{1}{3} \alpha_1 \dot{\omega}_1 l_1^3 = \frac{1}{3} m_1 l_1^2 \dot{\omega}_1 \tag{C-5}$$

in conformance to the product of (angular acceleration) • (moment of inertia). The formulation for the second segment is more complex, not just due to more terms but also because fewer terms in the cross product vanish:

[§] Various vector identities and simplifications are inserted without explicit demonstration throughout this development, to avoid excess length in the presentation. It is worth noting that this type of "side analysis" with limited scope uses familiar intuitive concepts (such as summing moments derived from products of weight × lever arm), whereas the formalism in the preceding Appendix instead relies on propagation of forces imparted via the matrix **P**.

$$\dot{\mathbf{H}}_2 = \alpha_2 \frac{d}{dt}\int_0^{l_2} (l_1\mathbf{u}_1 + s\mathbf{u}_2)\times(l_1\dot{\mathbf{u}}_1 + s\dot{\mathbf{u}}_2)\; ds$$

$$= \alpha_2\int_0^{l_2}(l_1\mathbf{u}_1+s\mathbf{u}_2)\times[\,l_1(\dot{\omega}_1\times\mathbf{u}_1-\omega_1^2\mathbf{u}_1)+s(\dot{\omega}_2\times\mathbf{u}_2-\omega_2^2\mathbf{u}_2)\,]\; ds$$

$$= \alpha_2\int_0^{l_2}\left\{l_1^2\dot{\omega}_1 + l_1 s\,[\,\mathbf{u}_1\cdot\mathbf{u}_2\,(\dot{\omega}_1+\dot{\omega}_2)+(\omega_1^2-\omega_2^2)\,\mathbf{u}_1\times\mathbf{u}_2\,]+s^2\dot{\omega}_2\right\} ds \qquad \text{(C-6)}$$

$$= m_2\left\{l_1^2\dot{\omega}_1 + \frac{1}{2}\,l_1 l_2\,[\,(\mathbf{u}_1\cdot\mathbf{u}_2)\,(\dot{\omega}_1+\dot{\omega}_2)+(\omega_1^2-\omega_2^2)\,\mathbf{u}_1\times\mathbf{u}_2\,]+\frac{1}{3}\,l_2^2\,\dot{\omega}_2\right\}$$

Since all torques and angular momentum components are parallel to the z-axis, for example

$$\mathbf{H}_2 = \mathbf{1}_3^{\mathrm{T}} H_2 \qquad \text{(C-7)}$$

the magnitude in Eq. (C-6) can be written as[§]

$$H_2 = m_2\left\{l_1^2\dot{w}_1 + \frac{1}{2}\,l_1 l_2\,[\,(\mathbf{u}_1\cdot\mathbf{u}_2)\,(\dot{w}_1+\dot{w}_2)+(w_1^2-w_2^2)\sin(\beta_2-\beta_1)\,]+\frac{1}{3}\,l_2^2\,\dot{w}_2\right\} \qquad \text{(C-8)}$$

where the w-elements are z-components of angular rate, as in Eq. (B-21). Rather than repeat this process with increasingly larger vector sums from Eq. (C-2) a computational approach is established. Each integral in Eq. (C-1) contains the following contributions:

- a term of the form $^1/_3\, m_i l_i^2 w_i$;
- a sum of terms, over all $j < i$ and $i,j \neq 4$, of the form $m_i l_j^2 w_j$ (without the $^1/_3$);
- a sum of terms, over all $j < i$ and $i,j \neq 4$, of the form

$$\pm\frac{1}{2}\,m_i\, l_i l_j\,[\,(\mathbf{u}_i\cdot\mathbf{u}_j)\,(\dot{w}_i+\dot{w}_j)+(w_i^2-w_j^2)\sin(\beta_j-\beta_i)\,] \qquad \text{(C-9)}$$

with a + sign if i and j are on the same side (right or left), and negative otherwise;

and • a double sum of terms, over all combinations $k < j < i$ and $\neq 4$, with the same form and sign criteria above, but with i replaced by k and without the $^1/_2$;

$$\pm\, m_i\, l_k l_j\,[\,(\mathbf{u}_k\cdot\mathbf{u}_j)\,(\dot{w}_k+\dot{w}_j)+(w_k^2-w_j^2)\sin(\beta_j-\beta_k)\,] \qquad \text{(C-10)}$$

Subroutine FRACEQ (APPENDIX D) sums these terms in double precision and equates net torque about the inertial reference point to the time derivative of angular momentum {which, per discussion following Eq. (2), contains weight-derived moments + (1-f)•(y-component of the total restoring force)•(horizontal lever arm distance between left and right foot)}. Again the left/right half-cycle symmetry mentioned in APPENDIX A is exploited, so that the sequence of values is computed for one foot only.

[§] As a special case test let $w_1\mathbf{u}_1 = w_2\mathbf{u}_2$ and note that Eq. (C-8) produces a moment of inertia equal to $m_2\,(\,l_1^2 + l_1 l_2 + l_2^2/3\,) = m_2\,[\,(\,l_1 + l_2/2\,)^2 + l_2^2/12\,]$, in agreement with the parallel axis theorem.

APPENDIX D: PROGRAM LISTINGS

```
      PROGRAM HIPROG

      IMPLICIT NONE
      INCLUDE 'ANGARR.INC'
      INTEGER N

      CALL INITC
      CALL ANGDAT
      DO N = 1,NPCYCL
        NP=N
        CALL KINEM
        CALL RODYN
        CALL TABUL
      END DO
      CALL PRNTBL
      CALL GRAPH
      END
```

```
      SUBROUTINE ANGDAT
      IMPLICIT NONE
      INCLUDE 'ANGARR.INC'
      INCLUDE 'DECL.INC'
      REAL*4 ANGH(100),ANGK(100)

      PI=4.*ATAN(1.)
      OPEN(UNIT=16,FILE='ANG.DAT',STATUS='OLD',FORM=
     *                     'FORMATTED',ACCESS='SEQUENTIAL')
      DO I=1,NPCYCL
        READ (16,9300,END=1000) ANGH(I),ANGK(I)
 1000         CONTINUE
        ANGHIP(I)=(PI/180.)*ANGH(I)
        ANGKNE(I)=(PI/180.)*ANGK(I)
      END DO
      CLOSE (UNIT=16)
 9300 FORMAT (2(1X,F8.3))
      END
```

```
      SUBROUTINE FSOLU
      IMPLICIT NONE
      INCLUDE 'ANGARR.INC'
      INCLUDE 'DECL.INC'
      INCLUDE 'ROTAT.INC'

      CALL LOAD2(RESTOR,0.,TOTALM*GRAV)
      DO I = 1,7
        CALL VECADD(2,RESTOR,WEIGHT(I)/GRAV,D2R(1,I),RESTOR)
      END DO
      IF (NP.LT.12) THEN
        CALL FRACEQ
      ELSE IF (NP.LT.51) THEN
        FRAC(NP) = 1.
      ELSE IF (NP.LT.62) THEN
        FRAC(NP) = 1. - FRAC(NP-50)
      ELSE
        FRAC(NP) = 0.
      END IF
      RETURN
      END
```

```
SUBROUTINE INITC
IMPLICIT NONE
INCLUDE 'ANGARR.INC'
INCLUDE 'DECL.INC'
INCLUDE 'OUTPUT.INC'
INCLUDE 'ROTAT.INC'
      NPCYCL = 100
      PRINT *, '# of cycles per min (1 cycle = 2 steps)'
      READ (*,*) NCYPM
      TAU = 60./(NCYPM*NPCYCL)                    ! 1 cycle = 2 steps
      NF = 15
      GRAV = 32.2
DO I = 1,100
  DO J = 1,7
          IF (I.LE.2) D2R(I,J) = 0.
          IF (J.LE.4) TAPPRX = 1.E+9
          IF (I.LE.6) RWT(I,J) = 0.
          IF (I.LE.7.AND.J.LE.6) RWC(I,J) = 0.
          IF (I.LE.7) RWMM (I,J) = 0.
          IF (I.LE.7) RWMU (I,J) = 0.
        END DO
END DO
CALL LOAD2(RWT(2,1),1.,1.)
CALL LOAD2(RWT(4,7),1.,1.)
RWT(3,2)=1.
RWT(4,6)=1.
      CALL LOAD7(RWL,0.5,2.,1.5,2.5,1.5,2.,0.5)
      CALL LOAD7(WEIGHT,3.,5.,12.,140.,12.,5.,3.)
DO I = 1,7
  TOTALM=TOTALM+WEIGHT(I)/GRAV
END DO
DO I = 1,7
  RWIZZ(I)=WEIGHT(I)*RWL(I)**2/(12.*GRAV)
  RWMM(I,I)=WEIGHT(I)/GRAV
  RWMU(I,I)=1.
  IF (I.LT.4) RWT(I,I)=1.
  IF (I.GT.4) RWT(I-1,I)=1.
  IF (I.LE.6) THEN
    RWC(I,I)=0.5*RWL(I)
    RWC(I+1,I)=0.5*RWL(I+1)
  END IF
  DO J=1,7
    RWMU(I,J)=RWMU(I,J)-WEIGHT(I)/(TOTALM*GRAV)
  END DO
END DO
CALL MTXMPY(7,7,6,0,RWC,RWT ,RWD)
CALL MTXMPY(7,7,7,0,RWD,RWMU,RWB)
CALL MTXMPY(7,7,7,0,RWB,RWMM,RWD)       !  write over orig. RWD : OK
CALL MTXMPY(7,7,7,2,RWD,RWB ,RWJ)       !  Only diag elememts are used
DO I = 1,7
  RWAUGI(I,I)=RWIZZ(I)+RWJ(I,I)
END DO
RETURN
END
```

```
SUBROUTINE BSOLU
IMPLICIT NONE
INCLUDE 'ANGARR.INC'
INCLUDE 'DECL.INC'
INCLUDE 'ROTAT.INC'
INTEGER NH
REAL*4 BINI,BFIN,COSB,T1,T2
REAL*4 TILT
IF (NP.EQ.1) THEN
  PRINT *, 'Tilt of Trunk in degrees ( ccw = + )'
  READ (*,*) TILT  ! back ERECT @0., FWD if TILT < 0. ( ccw = + )
  BETA(4)=TILT*PI/180. ! code, as is, covers only the constant tilt case
END IF
NH = 1+MOD(NP-1+NPCYCL/2,NPCYCL)
BETA(3) = BETA(4) + ANGHIP(NP)
BETA(5) = BETA(4) + ANGHIP(NH)
BETA(2) = BETA(3) - ANGKNE(NP)
BETA(6) = BETA(5) - ANGKNE(NH)
IF (NP.LT.12) THEN
  BETA(1) = PI/2.
  T1 = RWL(3) * (COS(BETA(3)) - COS(BETA(5)))
  T2 = RWL(2) * (COS(BETA(2)) - COS(BETA(6)))
  COSB = (T1 + T2) / RWL(1)
  T1 = ACOS(ABS(COSB))
  BETA(7) = T1
  IF (COSB.LT.0) THEN
    BETA(1) = T1
    BETA(7) = PI/2.
  END IF
  IF (NP.EQ.1)  BINI = BETA(7)
  IF (NP.EQ.11) BFIN = BETA(7)
ELSE IF (NP.LT.51) THEN
  IF (NP.LT.31) THEN
    BETA(1) = PI/2. - (PI/2.-BINI)*(NP-11)**2/800.
    BETA(7) = BFIN  + (PI/2.-BFIN)*(NP-11)**2/800.
  ELSE
    BETA(1) = BINI  + (PI/2.-BINI)*(51-NP)**2/800.
    BETA(7) = PI/2. - (PI/2.-BFIN)*(51-NP)**2/800.
  END IF
ELSE IF (NP.LT.62) THEN
  BETA(7) = PI/2.
  T1 = RWL(3) * (COS(BETA(5)) - COS(BETA(3)))
  T2 = RWL(2) * (COS(BETA(6)) - COS(BETA(2)))
  COSB = (T1 + T2) / RWL(1)
  T1 = ACOS(ABS(COSB))
  BETA(1) = T1
  IF (COSB.LT.0) THEN
    BETA(7) = T1
    BETA(1) = PI/2.
  END IF
ELSE
  IF (NP.LT.81) THEN
    BETA(7) = PI/2. - (PI/2.-BINI)*(NP-61)**2/800.
    BETA(1) = BFIN  + (PI/2.-BFIN)*(NP-61)**2/800.
  ELSE
    BETA(7) = BINI  + (PI/2.-BINI)*(101-NP)**2/800.
    BETA(1) = PI/2. - (PI/2.-BFIN)*(101-NP)**2/800.
  END IF
END IF
RETURN
END
```

```
      SUBROUTINE FRACEQ
      IMPLICIT NONE
      INCLUDE 'ANGARR.INC'
      INCLUDE 'DECL.INC'
      INCLUDE 'ROTAT.INC'
      INTEGER IM1,JM1,K
      DOUBLE PRECISION ARM,G(7),SGN,GAMMA,UU(2,7),RS(7),Y(7),Z(6,7),DDOT
*      Y(I) = DBLE(RWL(I))**2*DBLE(WDOT(I))
*      Z(I,J)=(DDOT(2,UU(1,I),UU(1,J))*(DBLE((WDOT(I))+DBLE(WDOT(J))))
*     ^ + (DBLE(OMEGA(I))**2-DBLE(OMEGA(J))**2)*SIN(BETA(J)-BETA(I)))
*     ^ *  DBLE(RWL(I))*DBLE(RWL(J))
*      RS(I) = DBLE(RWL(I))*DSIN(DBLE(BETA(I)))
      GAMMA = 0.D0
      ARM = 0.D0
      DO I=1,7
        UU(1,I) = - DSIN(DBLE(BETA(I)))
        UU(2,I) = DCOS(DBLE(BETA(I)))
      END DO

      DO I=1,7
       IF (I.NE.4) THEN
        Y(I) = DBLE(RWL(I))**2*DBLE(WDOT(I))
        G(I) = 0.3333D0*Y(I)
        IM1  = I-1
        DO J=1,IM1
          JM1  = J-1
          IF (J.NE.4.AND.J.LT.I) THEN
            Y(J) = DBLE(RWL(J))**2*DBLE(WDOT(J))
            G(I) = G(I) + Y(J)
            DO K=1,JM1
              SGN = 1.D0
              IF ((J-4)*(K-4).LT.0) SGN = -1.D0
              Z(K,J) = (DDOT(2,UU(1,K),UU(1,J)) * (DBLE((WDOT(K)) +
     ^        DBLE(WDOT(J)))) + (DBLE(OMEGA(K))**2-DBLE(OMEGA(J))**2) *
     ^        SIN(BETA(J)-BETA(K))) *  DBLE(RWL(K))*DBLE(RWL(J))
              IF (K.NE.4.AND.K.LT.J) G(I) = G(I) + SGN*Z(K,J)
            END DO                                                   ! K
            SGN = 1.D0
            IF ((J-4)*(I-4).LT.0) SGN = -1.D0
            Z(J,I) = (DDOT(2,UU(1,J),UU(1,I)) * (DBLE((WDOT(J)) +
     ^      DBLE(WDOT(I)))) + (DBLE(OMEGA(J))**2-DBLE(OMEGA(I))**2) *
     ^      SIN(BETA(I)-BETA(J))) *  DBLE(RWL(J))*DBLE(RWL(I))
            G(I) = G(I) + 0.5D0*SGN*Z(J,I)
          END IF
        END DO                                                       ! J
        G(I) = (DBLE(WEIGHT(I))/DBLE(GRAV)) * G(I)
        SGN = 1.D0
        IF (I.EQ.1) SGN = -1.D0   ! for 1 to 12 period ONLY; neg @I=7 for 51 ...
        RS(I) = DBLE(RWL(I))*DSIN(DBLE(BETA(I)))
        IF (I.NE.1) ARM = ARM + SGN*DABS(RS(I))
        GAMMA = GAMMA + DBLE(WEIGHT(I))*SGN*DABS(RS(I)) - G(I)
       END IF
      END DO                                                         ! I

      FRAC(NP) = SNGL( 1.D0 - GAMMA / ( ARM*DBLE(RESTOR(2))) )
      RETURN
      END
```

```
SUBROUTINE KINEM
 IMPLICIT NONE
 INCLUDE 'ANGARR.INC'
 INCLUDE 'DECL.INC'
 INCLUDE 'ROTAT.INC'
 INTEGER IENTRY                    /0/
 INTEGER K,MID,MODU,LAST,NEXT            ! & I0 for RO.DAT file
 REAL*4 B4PREV,W4PREV,RATEHP,RATEKN
       DOUBLE PRECISION Z
       Z(I,J) = DBLE(RWL(J)*D2U(I,J))
 IF (IENTRY.EQ.0) THEN
   IF (MOD(NPCYCL,2).NE.0) STOP
   IENTRY=IENTRY+1
 END IF
 OMEGA(4)=(BETA(4)-B4PREV)/TAU          ! strictly
 IF (IENTRY.EQ.0) OMEGA(4) = 0.         ! correct
 B4PREV=BETA(4)                         ! only
 WDOT(4)=(OMEGA(4)-W4PREV)/TAU          ! for
 IF (IENTRY.EQ.0) WDOT(4) = 0.          ! constant
 W4PREV=OMEGA(4)                        ! beta(4)
       CALL BSOLU
 DO I=1,2
   MODU=NPCYCL/2+(MOD(I,2))*NPCYCL/2
   LAST=1+MOD(NP-2+MODU,NPCYCL)
   NEXT=1+MOD(NP  +MODU,NPCYCL)
   RATEHP=(ANGHIP(NEXT)-ANGHIP(LAST))/(2.*TAU)
   RATEKN=(ANGKNE(NEXT)-ANGKNE(LAST))/(2.*TAU)
   OMEGA(2*I+1)=OMEGA(4)+RATEHP
   OMEGA(2*I+1+(-1)**I)=OMEGA(2*I+1)-RATEKN
   MID=1+MOD(LAST,NPCYCL)
   WDOT(2*I+1)=WDOT(4)+(ANGHIP(NEXT)-
    ^                2.*ANGHIP(MID)+ANGHIP(LAST))/TAU**2
   WDOT(2*I+1+(-1)**I)=WDOT(2*I+1)-(ANGKNE(NEXT)-
    ^                2.*ANGKNE(MID)+ANGKNE(LAST))/TAU**2
 END DO
 DO I = 1,7     ! OMEGA(1),WDOT(1) & OMEGA(7),WDOT(7) are zero here
   D2U(1,I)=-WDOT(I)*COS(BETA(I))+OMEGA(I)**2*SIN(BETA(I))
   D2U(2,I)=-WDOT(I)*SIN(BETA(I))-OMEGA(I)**2*COS(BETA(I))
 END DO
       CALL FSOLU
 DO I = 1,2                  ! Z(I,J) = DBLE(RWL(J)*D2U(I,J)) :
  IF (FRAC(NP).GT.0.5.OR.NP.LE.2) THEN
   D2R(I,1)=SNGL(.5D0*Z(I,1))
   D2R(I,2)=SNGL(Z(I,1)+.5D0*Z(I,2))
   D2R(I,3)=SNGL(Z(I,1)+Z(I,2)+.5D0*Z(I,3))
   D2R(I,4)=SNGL(Z(I,1)+Z(I,2)+Z(I,3)+.5D0*Z(I,4))
   D2R(I,5)=SNGL(Z(I,1)+Z(I,2)+Z(I,3)-.5D0*Z(I,5))
   D2R(I,6)=SNGL(Z(I,1)+Z(I,2)+Z(I,3)-Z(I,5)-.5D0*Z(I,6))
   D2R(I,7)=SNGL(Z(I,1)+Z(I,2)+Z(I,3)-Z(I,5)-Z(I,6)-.5D0*Z(I,7))
  ELSE
   D2R(I,7)=SNGL(.5D0*Z(I,7))
   D2R(I,6)=SNGL(Z(I,7)+.5D0*Z(I,6))
   D2R(I,5)=SNGL(Z(I,7)+Z(I,6)+.5D0*Z(I,5))
   D2R(I,4)=SNGL(Z(I,7)+Z(I,6)+Z(I,5)+.5D0*Z(I,4))
   D2R(I,3)=SNGL(Z(I,7)+Z(I,6)+Z(I,5)-.5D0*Z(I,3))
   D2R(I,2)=SNGL(Z(I,7)+Z(I,6)+Z(I,5)-Z(I,3)-.5D0*Z(I,2))
   D2R(I,1)=SNGL(Z(I,7)+Z(I,6)+Z(I,5)-Z(I,3)-Z(I,2)-.5D0*Z(I,1))
   END IF
 END DO
 RETURN
 END
```

```
   SUBROUTINE RODYN
   IMPLICIT NONE
   INCLUDE 'ANGARR.INC'
   INCLUDE 'ROTAT.INC'
   INCLUDE 'DECL.INC'
   INCLUDE 'OUTPUT.INC'
         REAL*4 FRIC,CB1,SB1,CB7,SB7,VEC1(7)

   FRIC = 0.
   CALL FSOLU
         CB1 = COS(BETA(1))
         SB1 = SIN(BETA(1))
         CB7 = COS(BETA(7))
         SB7 = SIN(BETA(7))
   DO I = 1,7
*RWP(I,1)=-RWB(I,1)*COS(BETA(I)-BETA(1)) RWP(I,2)=-RWB(I,1)*SIN(BETA(I)-BETA(1))
*RWP(I,3)=-RWB(I,7)*COS(BETA(I)-BETA(7)) RWP(I,4)=-RWB(I,7)*SIN(BETA(I)-BETA(7))
           RWPF(I) = - RWB(I,1) * (     RESTOR(1) * COS(BETA(I)) +
     ^                                  RESTOR(2) * SIN(BETA(I)) )
     ^                - RWB(I,7) * (     RESTOR(1) * COS(BETA(I)) +
     ^                                  RESTOR(2) * SIN(BETA(I)) )
   END DO
   DO I = 1,7
     RWQ(I)=0.
*    SPTORQ(I)=0.
     DO J=1,7
       IF (I.NE.J) THEN
        RWAUGI(I,J)=-TOTALM*RWB(I,J)*RWB(J,I)*COS(BETA(I)-BETA(J))
        RWQ(I)=RWQ(I)+TOTALM*RWB(I,J)*RWB(J,I)*OMEGA(J)**2*SIN
     ^                                             (BETA(I)-BETA(J))
       END IF
     END DO
   END DO
   CALL MTXMPY(7,1,7,0,RWAUGI,WDOT,VEC1)
   DO I = 1,7
     TORQ7(I)=VEC1(I)+RWPF(I)-RWQ(I)       ! & SPTROQ(...)
         END DO
         IF (BETA(1).LT.1.57) TORQ7(1) = TORQ7(1) + 0.5*RWL(1)*
     ^               FRAC(NP) * ( (-SB1)*RESTOR(2) - (CB1)*RESTOR(1) )
         IF (BETA(7).LT.1.57) TORQ7(7) = TORQ7(7) + 0.5*RWL(7)*
     ^        (1. - FRAC(NP)) * ( (-SB7)*RESTOR(2) - (CB7)*RESTOR(1) )
   CALL MTXMPY(6,1,7,0,RWT,TORQ7,HTORQ)
*         SPTORQ(2) = SPK*(ANGKNE(NP)-REFKNE)
*         SPTORQ(3) = ...    SPTORQ(4) = ...    SPTORQ(5) = ...
*         subtract: MTORQ = HTORQ - SPTORQ
   IF (NP.GT.61) THEN
     TAPPRX(NP,1)=(WEIGHT(1)+0.5*WEIGHT(2))*RWL(2)*SIN(BETA(2))
     TAPPRX(NP,2)=WEIGHT(3)*0.5*RWL(3)*SIN(BETA(3)) +
     ^         WEIGHT(2)*(RWL(3)*SIN(BETA(3))+0.5*RWL(2)*SIN(BETA(2))) +
     ^         WEIGHT(1)*(RWL(3)*SIN(BETA(3))  +  RWL(2)*SIN(BETA(2)))
   END IF
   IF (NP.GT.11.AND.NP.LT.51) THEN
     TAPPRX(NP,4)=(WEIGHT(7)+0.5*WEIGHT(6))*RWL(6)*SIN(BETA(6))
     TAPPRX(NP,3)=WEIGHT(5)*0.5*RWL(5)*SIN(BETA(5)) +
     ^         WEIGHT(6)*(RWL(5)*SIN(BETA(5))+0.5*RWL(6)*SIN(BETA(6))) +
     ^         WEIGHT(7)*(RWL(5)*SIN(BETA(5))  +  RWL(6)*SIN(BETA(6)))
   END IF
   RETURN
   END
```

```
      SUBROUTINE TABUL
      IMPLICIT NONE
      INCLUDE 'ANGARR.INC'
      INCLUDE 'DECL.INC'
      INCLUDE 'ROTAT.INC'
      INCLUDE 'OUTPUT.INC'

        TAFRAC(NP)=FRAC(NP)
        TANGH(NP)=(180./PI)*ANGHIP(NP)
        TANGK(NP)=(180./PI)*ANGKNE(NP)
        TABETA(NP,6)=(180./PI)*BETA(7)
      DO I=1,5
        TABETA(NP,I)=(180./PI)*BETA(I)
        IF (I.GE.4) TABETA(NP,I)=(180./PI)*BETA(I+1)
*       TAOMEG(NP,I)=(180./PI)*TAU*OMEGA(I+1)
*       TAWDOT(NP,I)=(180./PI)*TAU**2*WDOT(I+1)
*       DO J = 1,2
*         TAD2U(NP,J,I)=TAU**2*D2U(J,I+1)
*         TAD2R(NP,J,I)=TAU**2*D2R(J,I+1)
*       END DO
        TARWPF(NP,I)=RWPF(I+1)
*       TAUGWD(NP,I)=AUGIWD(I+1)
        TARWQ(NP,I)=RWQ(I+1)
*       TATOQ5(NP,I)=TORQ7(I+1)
      END DO
      DO J = 1,4
          TAHTOQ(NP,J)=HTORQ(J+1)
          IF (J.LE.2) TARSTR(NP,J)=RESTOR(J)
      END DO
      RETURN
      END
```

```
      SUBROUTINE PRNTBL
      IMPLICIT NONE
      INCLUDE 'DECL.INC'
      INCLUDE 'OUTPUT.INC'
      INCLUDE 'ROTAT.INC'
      INTEGER N
      REAL*4 B4deg

      OPEN(UNIT=NF,FILE='PARTAB.DAT',STATUS='NEW',FORM=
     *                    'FORMATTED',ACCESS='SEQUENTIAL')
      B4deg = 180./PI*BETA(4)
*      WRITE (NF,9000)
      DO N=1,100
  IF (N.EQ.1) WRITE(NF,9111) NCYPM,B4deg
  IF (N.EQ.1) WRITE(NF,9101)
  IF (N.EQ.51) WRITE(NF,9102)
  WRITE (NF,9100)N,TAFRAC(N),TANGH(N),TANGK(N),(TABETA(N,I),I=1,6)
  IF (MOD(N,10).EQ.0.AND.N.NE.50.AND.N.NE.100) WRITE(NF,9001)
      END DO

      DO N=1,100
  IF (N.EQ.1.OR.N.EQ.51) WRITE(NF,9601)
  WRITE (NF,9600) N,(TARWPF(N,I),I=1,5),(TARWQ(N,I),I=1,5)
  IF (MOD(N,10).EQ.0.AND.N.NE.50.AND.N.NE.100) WRITE(NF,9001)
      END DO

      DO N=1,100
  IF (N.EQ.1.OR.N.EQ.51) WRITE(NF,9701)
  IF (MOD(N,10).EQ.1.AND.N.NE.1.AND.N.NE.51) WRITE(NF,9001)
  WRITE (NF,9700,ERR=1000) N,(TARSTR(N,J),J=1,2),
     ^            (TAPPRX(N,J),J=1,4),(TAHTOQ(N,J),J=1,4)
 1000 CONTINUE
      END DO

      CLOSE (UNIT=NF)

      RETURN
* 9000 FORMAT (1H1/)
 9001 FORMAT (/)
 9111 FORMAT (1H1/12X,I3,' Cycles/min',13X,'Trunk ',F4.1
     *                   ' deg off Vertical')
 9101 FORMAT (6X,'frac',4X,'angh',4X,'angk',7X,'b1',6X,'b2',6X,'b3'
     *          ,7X,'b5',6X,'b6',6X,'b7'/)
 9102 FORMAT (1H1//6X,'frac',4X,'angh',4X,'angk',7X,'b1',6X,'b2',6X,'b3'
     *          ,7X,'b5',6X,'b6',6X,'b7'/)
 9100 FORMAT (1X,I3,1X,F5.3,2(1X,F7.1),2X,3(1X,F7.1),1X,3(1X,F7.1))
 9601 FORMAT (1H1/7X,'|<----------   [ P ] F   ---------->|<',
     *          '------------   Q   ------------>|'/)
 9600 FORMAT (1X,I3,2X,5(1X,F6.1),2X,5(1X,F6.1))
 9701 FORMAT (1H1/6X,'|<- Restor ->|<--------  Approx.  -------->|<----'
     *          ,'  Hinge Torques  ---->|'/)
 9700 FORMAT (1X,I3,2(1X,F6.1),2(2X,4(1X,F6.1)))
      END
```